CAP representations

(The mathematical theory of pyramidal algorithms)

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Wavelet Theory and Applications: Singapore, August 2004

Outline

- The pyramidal algorithm of Burt and Adelson
- Wavelet and framelet pyramids
- Function space characterizations via wavelets and framelets coefficients
- Approximation properties of framelets
- Compression-Alignment-Prediction (CAP) representations and their use in function space characterizations
- Compression-Alignment-Modified Prediction (CAMP) representations and their use in function space characterizations

Wisconsin

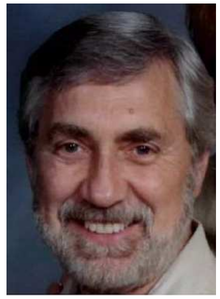
From left, 1st row:

Julia Velikina, Youngmi Hur, Yeon Kim, Narfi Stefansson.

2nd row:

Thomas Hangelbroek, Sangnam Nam, Jeff Kline, Steven Parker.

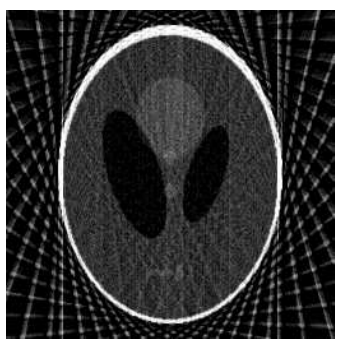
Julia Velikina: undersampled MRI data


Schepp−Logan phantom

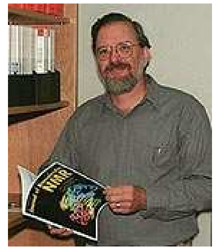
Conventional recon. from 90 projections, acceptable quality

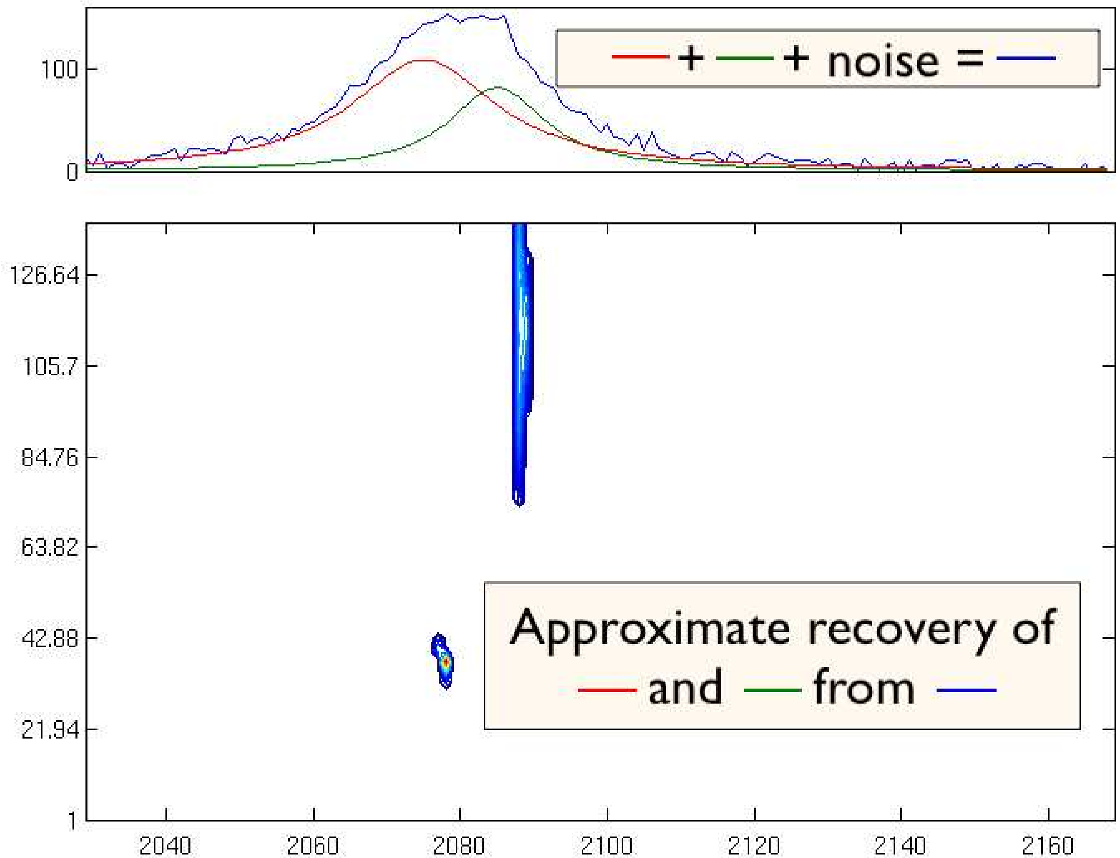
Conventional recon. from 23 projections, unacceptable quality



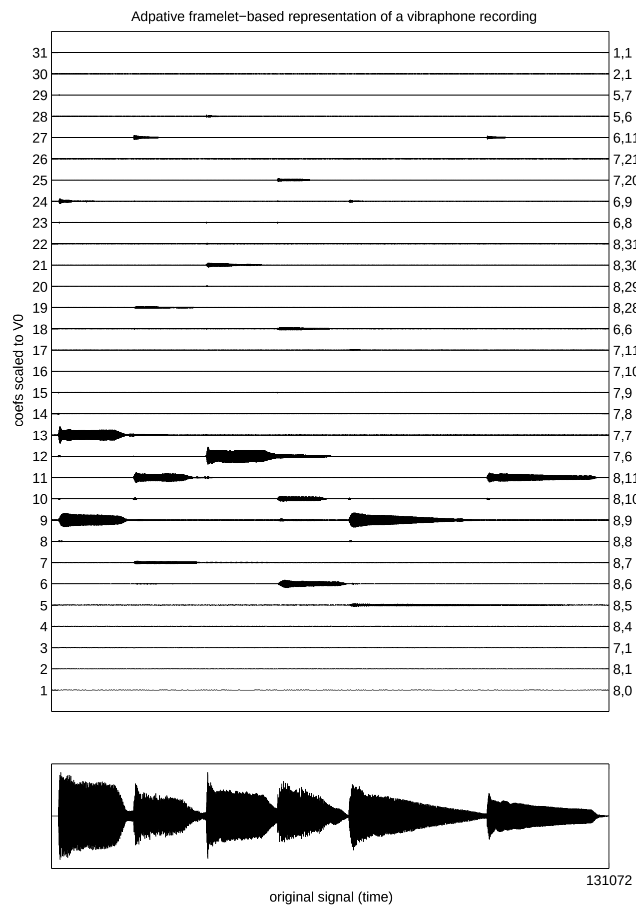
TV−based recon. from 23 projections

Jeff Kline: new data representation in NMR

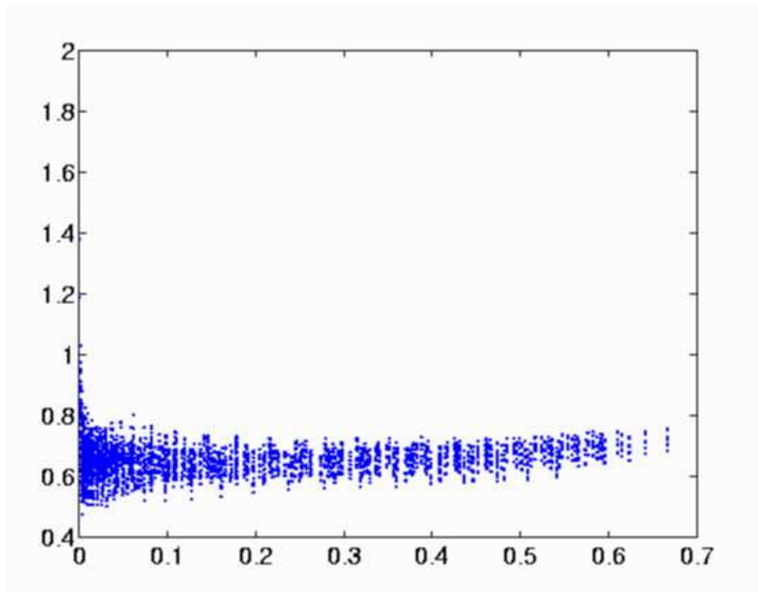




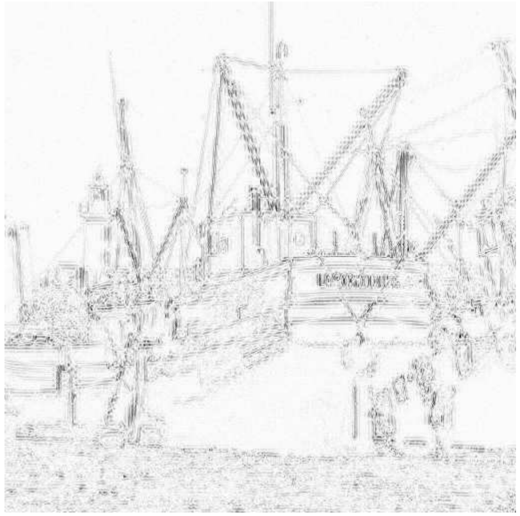
Steven Parker: redundant representation of acoustic signals



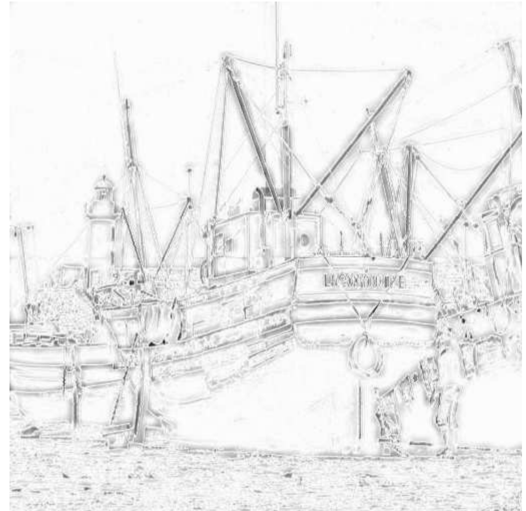
Narfi Stefansson: sprase framelet representations



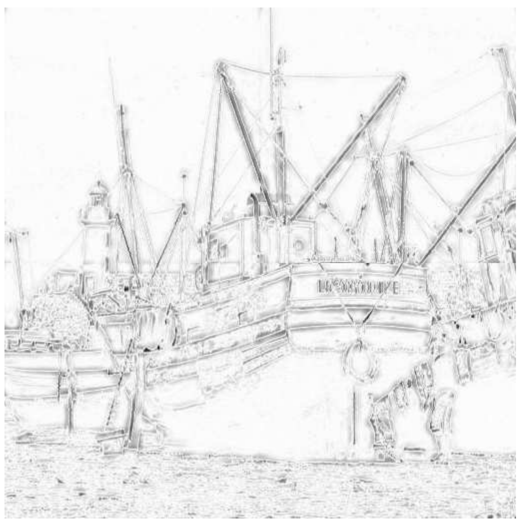
6/10 61440 coefficients



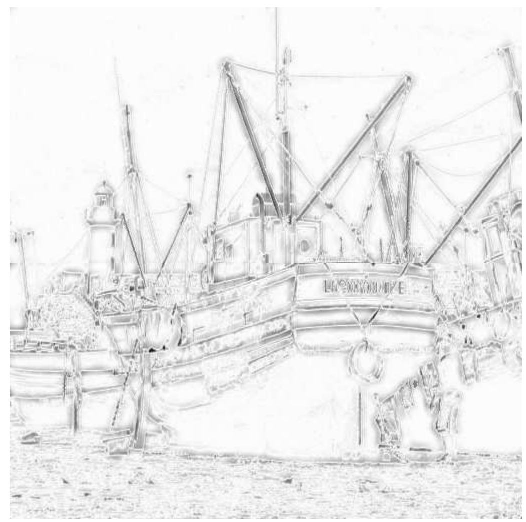
quartic spline 34452 coefficients



cubic spline 34608 coefficients



box15,box17,box18 35085 coefficients



FrameNet: on-line interactive framelet and wavelet analysis

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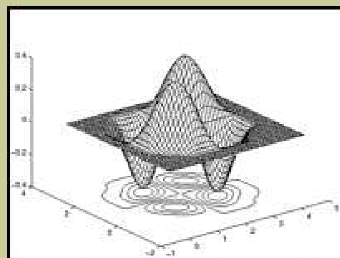


Version 1.0 Beta, October 2003

Welcome to **The IDR FrameNet Portal**, a web-based, research and educational tool for time/frequency analysis of data. If you are new to this site, we encourage you to take the tour or visit the Site Help.

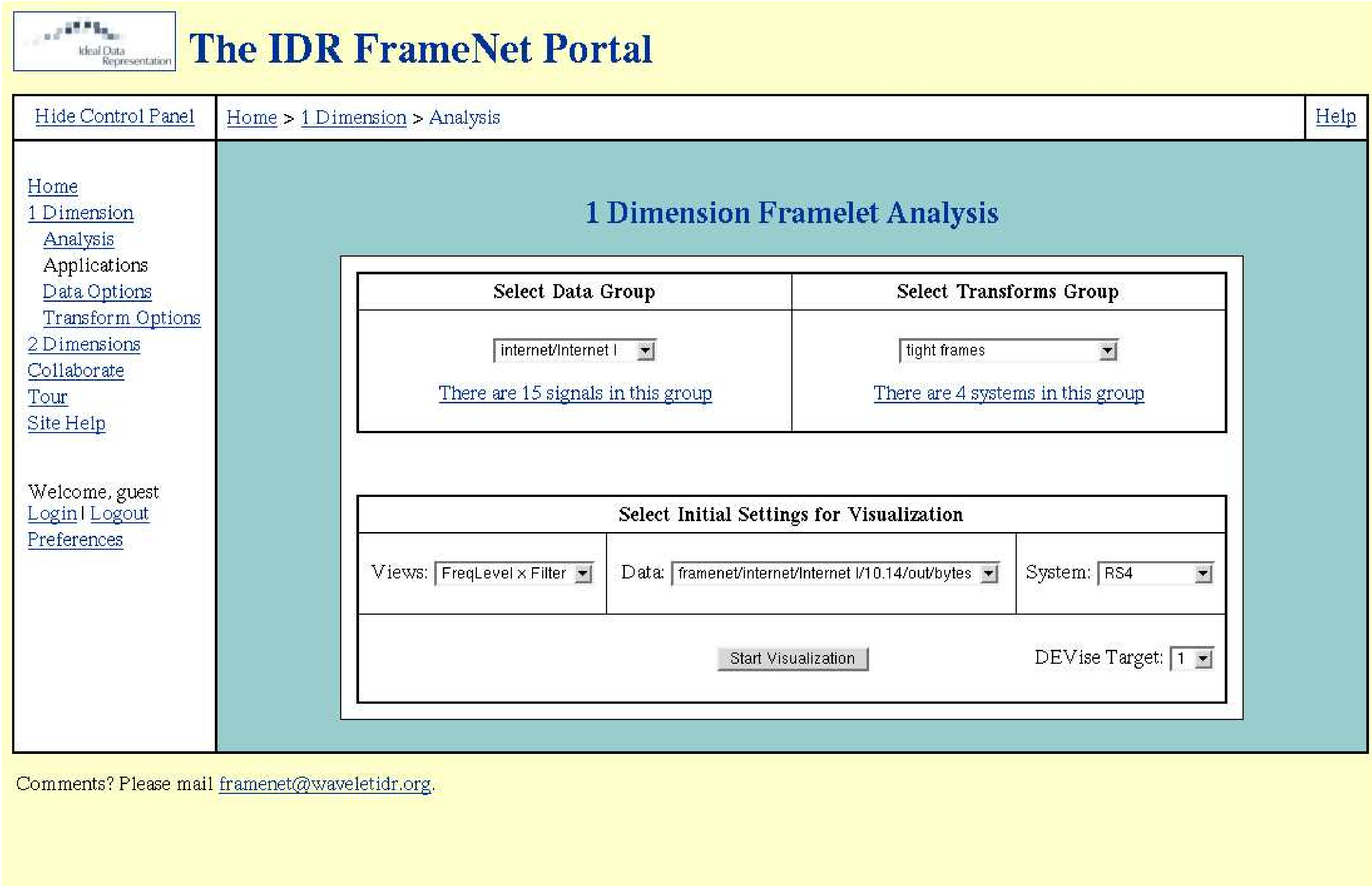
This tool provides facilities for uploading and management of scientific data, as well as dozens of available data sets from a variety of sources. Time/frequency analysis can be performed by classic wavelet systems, as well as by framelet systems (giving a redundant time/frequency description.) Furthermore, the FrameNet provides a collaboration mode, allowing researchers to work together on projects, and educators to demonstrate framelet analysis to their classes.

Group Leader: Amos Ron
Development Team Leader: Steven Parker
Development Team: Thomas Hangelbroek, Youngmi Hur, Jeff Kline, Narfi Stefansson, Bee-Chung Chen
with contributions from Carl de Boor, Miron Livny, Kent Wenger and Remi Gribonval.

This site is a project of the Wavelet Center for Ideal Data Representation. It incorporates the DEVise data exploration system and the LastWave signal processing software. Web hosting is maintained by Computer Systems Lab of the Computer Sciences Department, University of Wisconsin - Madison.

To contact the FrameNet team, send mail to framenet@waveletidr.org.

Comments? Please mail framenet@waveletidr.org.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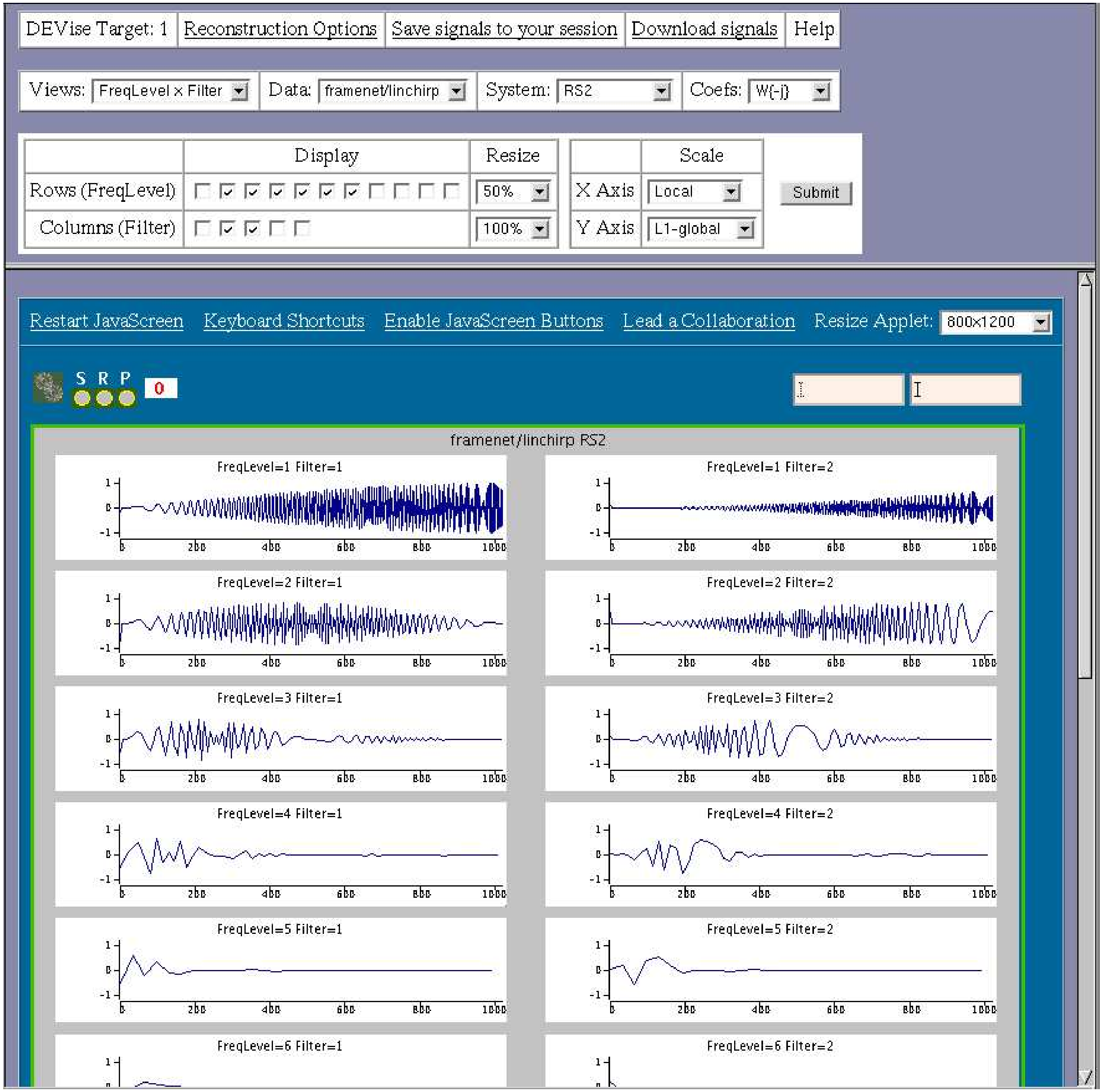

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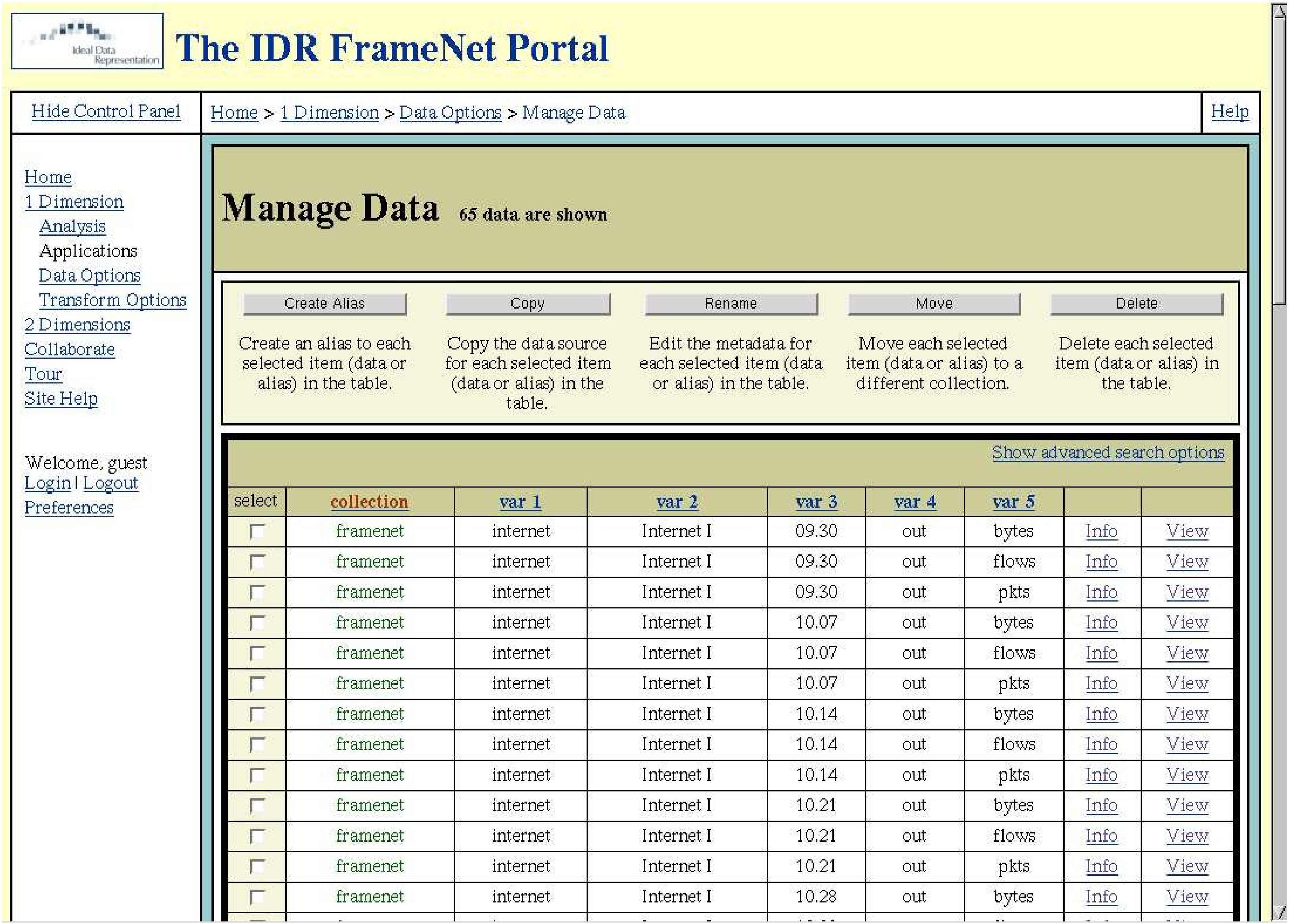
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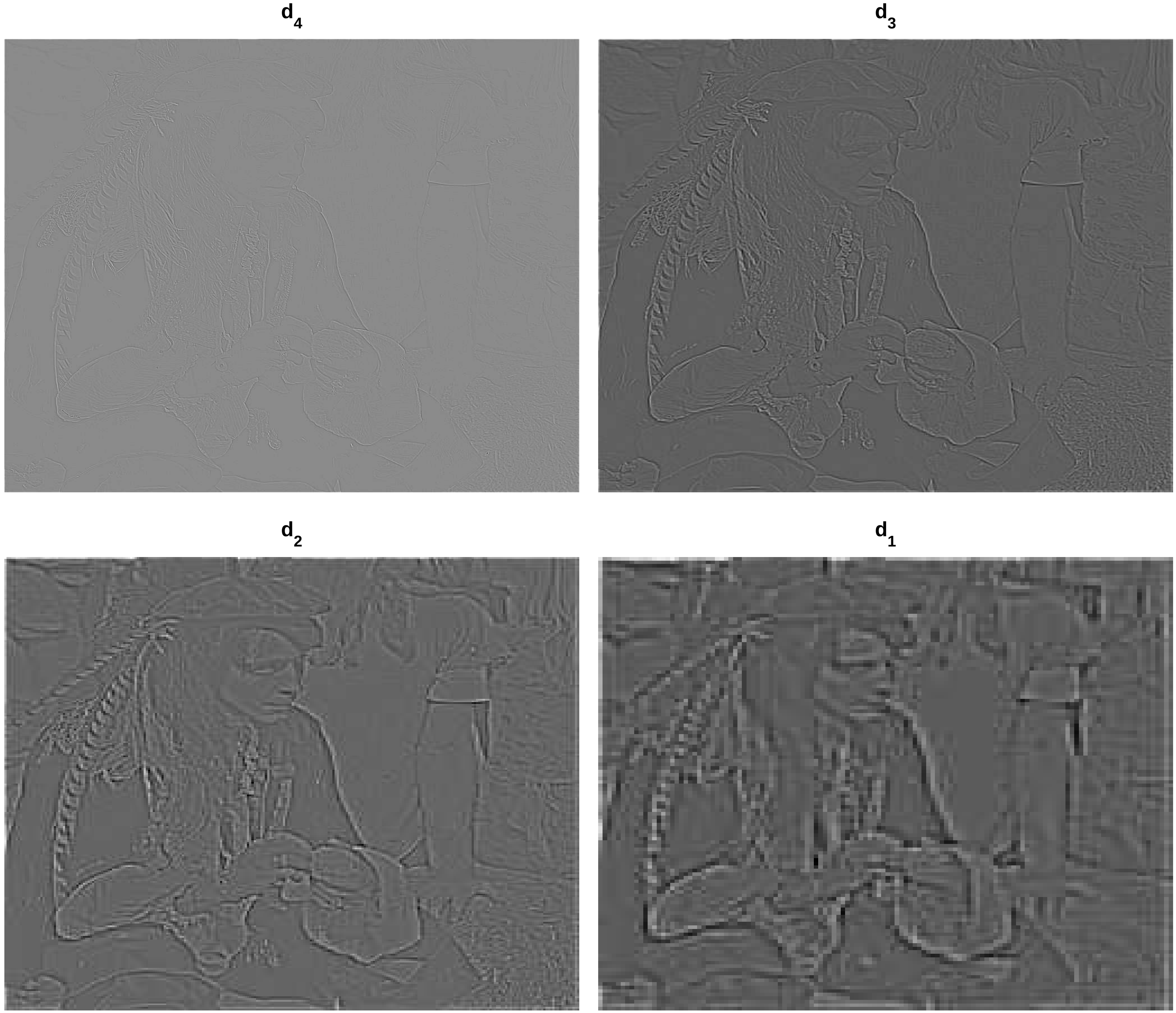


d_4
d_3
d_2
d_1

Pyramid algorithms (Burt and Adelson, 1983)

$h : \mathbb{Z} \to \mathbb{R}$ is a symmetric, normalized, filter:
$h(k) = h(-k)$, $\sum_{k\in\mathbb{Z}} h(k) = 1$.

$\downarrow, \uparrow$ are downsampling & upsampling:

$$\begin{aligned} y_{\downarrow}(k) &= y(2k), \quad k \in \mathbb{Z} \\ y_{\uparrow}(k) &= \begin{cases} 2y(k/2), & k \text{ even}, \\ 0, & \text{otherwise}. \end{cases} \end{aligned}$$

$(y_j)_{j=-\infty}^{\infty} \subset \mathbb{C}^{\mathbb{Z}}$ s.t:

$$y_j = C y_{j+1} := (h * y_j)_{\downarrow}, \quad \forall j.$$

C is Compression or Coarsification

---

y_{j+1} is then **predicted** from y_j by

$$y_{j+1} \approx P y_j := h * (y_{j\uparrow}).$$

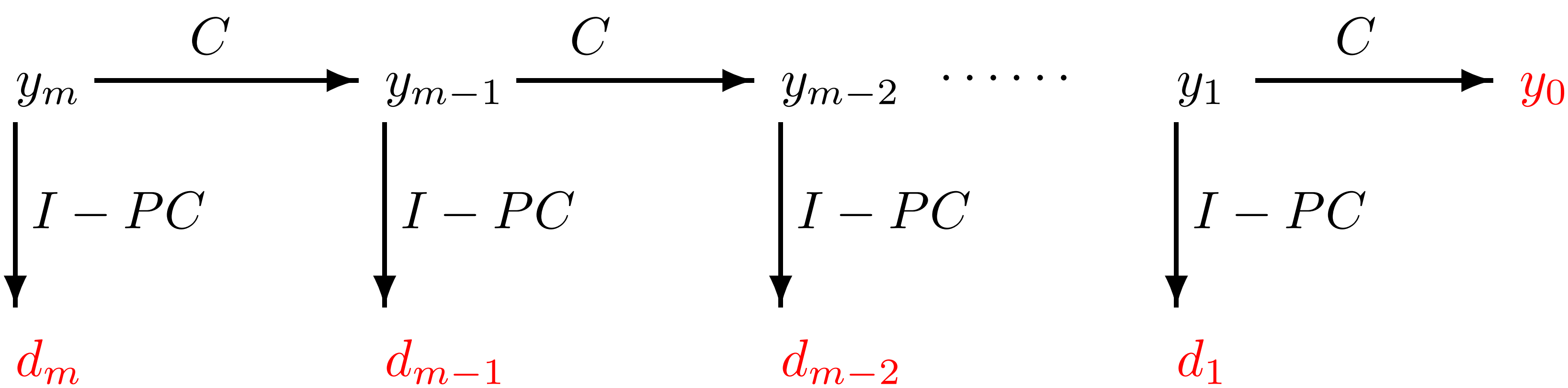
P is Prediction or subdivision

The **pyramid algorithm**:

- Define the detail coefficients:

$$d_j := (I - PC)\, y_j = y_j - P\, y_{j-1}.$$

- Replace y_j by the pair $(y_{j-1},\, d_j)$.
- Continue iteratively.



Reconstruction. Recovering y_m from $y_0, d_1, d_2, \ldots, d_m$ is trivial:

$y_1 = d_1 + P y_0$, $y_2 = d_2 + P y_1$ and so on.

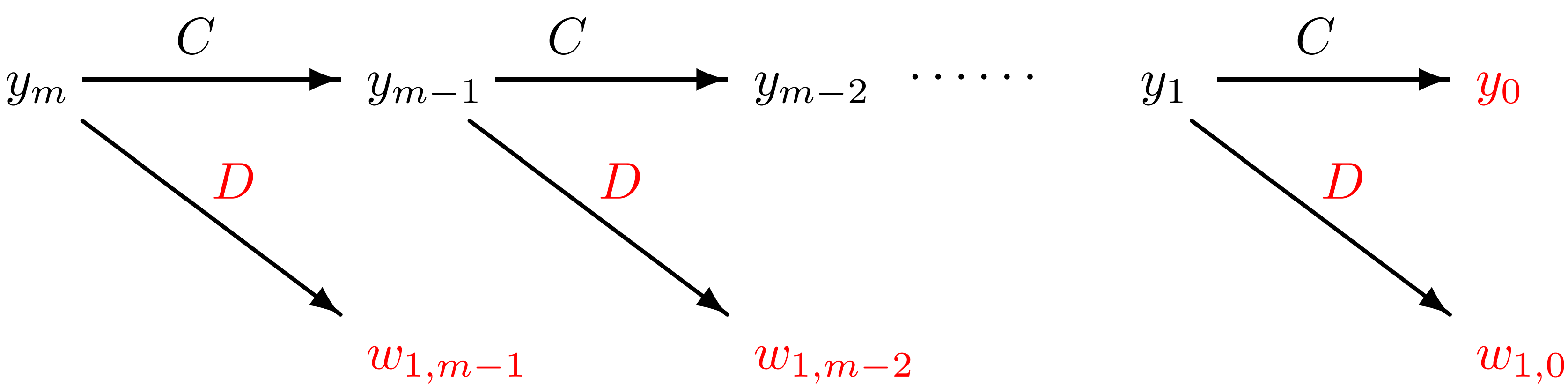
Wavelet pyramids, Mallat, 1987

Decompose the detail map $I - PC$:

$$I - PC = RD$$

$$D : y_j \mapsto (h_1 * y_j)_\downarrow =: w_{1,j-1}, \quad R : y \mapsto h_1 * y_\uparrow$$

with h_1 a real, symmetric, highpass: $\sum_{k \in \mathbb{Z}} h_1(k) = 0$.



Note that we can recover y_m from $y_0, w_{1,0}, w_{1,1}, \ldots, w_{1,m-1}$ since $y_1 = Rw_{1,0} + Py_0$, $y_2 = Rw_{1,1} + Py_1$ and so on.

Framelets pyramids, Daubechies-Han-R-Shen, 03

Decompose $I - PC = \sum_{i=1}^{r} R_i D_i$ where

$$D_i : y_j \mapsto (h_i * y_j)_{\downarrow} =: w_{i,j-1}, \quad R_i : y \mapsto h_i * (y_{j-1\uparrow})$$

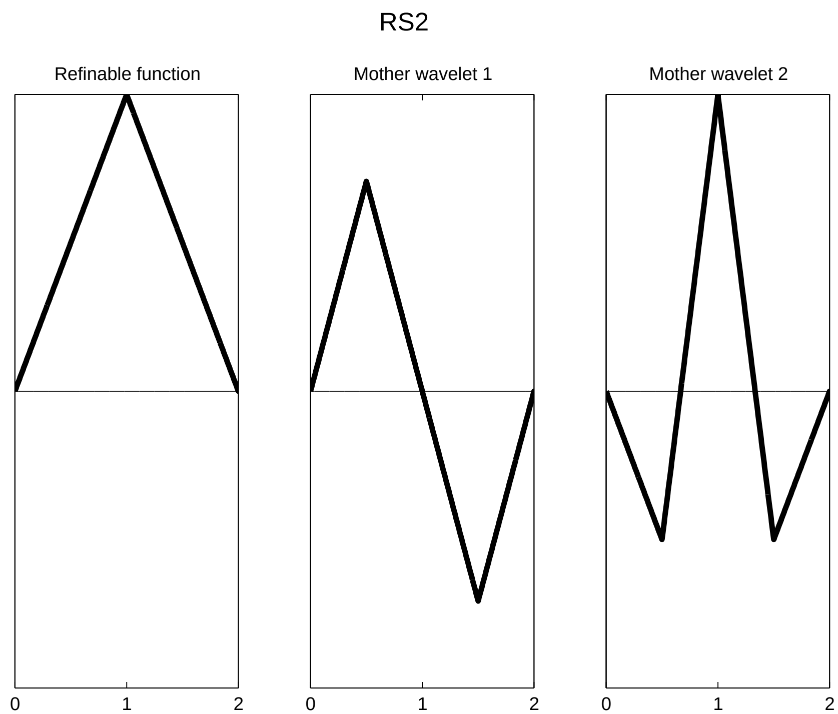
each h_i real, (anti-)symmetric, highpass: $\sum_{k \in \mathbb{Z}} h_i(k) = 0$.

Who needs the overhead associated with framelets?

A: they offer far greater design freedom

Example (R-Shen, 1997) :

$$\widehat{h}(\xi) = \frac{1}{4}\left(1+e^{-i\xi}\right)^2, \widehat{h_1}(\xi) = -\frac{\sqrt{2}}{4}\left(1-e^{-2i\xi}\right), \quad \widehat{h_2}(\xi) = -\frac{1}{4}\left(1-e^{-i\xi}\right)^2$$



piecewise-linear

All the filters above are 3-tap, and the underlying wavelets are piecewise linear

The mathematics behind pyramidal algorithms, Part I: the rudiments

Given h, one looks for $\phi \in L_2(\mathbb{R})$ s.t.

$$\widehat{\phi}(2\cdot) = \widehat{h}\widehat{\phi}, \quad (\hat{\phi}(0) = 1). \tag{1}$$

ϕ is a refinable function,
the filter h is the (lowpass) refinement mask.

Notation: For $j, k \in \mathbb{Z}$, $\varphi_{j,k} := 2^{j/2}\varphi(2^j \cdot -k)$. ($\varphi$ some function)

$$y_{j,f}(k) := 2^{j/2} \langle f, \phi_{j,k} \rangle.$$

Then $y_{j,f} = C y_{j+1,f}, \ \forall j.$

The mathematics behind pyramidal algorithms, Part II: wavelet-based characterizations of functions spaces

Let $\psi \in L_2(\mathbb{R}) \cap L_1(\mathbb{R})$ s.t. $\int \psi(t)dt = 0$

Wavelet system $X(\psi)$ is

$$X(\psi) := \left\{ \psi_{j,k} = 2^{j/2}\psi\left(2^j \cdot -k\right) : j,k \in \mathbb{Z} \right\}.$$

$X(\psi)$ is an **(orthonormal) wavelet** if it forms an orthonormal basis of $L_2(\mathbb{R})$.

Why wavelets ?

- fast algorithms
- function space characterizations
- non-redundant representations

Function spaces L_p^s $(s \in \mathbb{R}, 0 < p < \infty)$

Let $\varphi \in \mathcal{S}$ satisfy

$$\begin{aligned} &\operatorname{supp} \widehat{\varphi} \subset \{1/2 \leq |\xi| \leq 2\}, \\ &|\widehat{\varphi}(\xi)| \geq c > 0, \qquad 3/5 \leq |\xi| \leq 5/3, \\ &\textstyle\sum_{j \in \mathbb{Z}} |\hat{\varphi}(2^{-j}\xi)|^2 = 1, \qquad \xi \in \mathbb{R} \setminus \{0\}. \end{aligned}$$

The function space L_p^s is the set of all $f \in \mathcal{S}'/\mathcal{P}$ s.t.

$$\|f\|_{L_p^s} := \left\| \left(\sum_{j \in \mathbb{Z}} (2^{js} |\varphi_j * f|)^2 \right)^{1/2} \right\|_{L_p} < \infty, \quad \varphi_j := 2^j \varphi(2^j \cdot).$$

- $L_p^0 \approx L_p$ if $1 < p < \infty$
- $L_p^0 \approx H_p$ (the Hardy space) if $0 < p \leq 1$.
- $L_p^0 \cap L_p^m \approx W_p^m$ (the Sobolev space) if $1 < p < \infty$ and $m \in \mathbb{N}$.

Characterization of L_p^s using wavelets

Theorem 1 (Meyer, Frazier-Jawerth, 198x)
$n > \max\{s, -s, 1/p - 1 - s\}$, *integer.*
$X(\psi)$ *is orthonormal wavelet, and:*

$$\psi \in C_c^n, \qquad \int t^\alpha \psi(t) dt = 0, \quad \forall 0 \le \alpha \le n - 1.$$

Then we have

$$\|f\|_{L_p^s} \approx \|Q_\psi^s f\|_{L_p},$$

where

$$Q_\psi^s f := \left(\sum_{j,k} \left| \langle f, \psi_{j,k} \rangle \, 2^{js} \, \chi_{j,k} \right|^2 \right)^{1/2}, \quad \chi(t) := \begin{cases} 1, & 0 \le t < 1, \\ 0, & \textit{otherwise.} \end{cases}$$

Framelets

$\Psi \subset L_2$ is a finite set of mother wavelets

The affine system generated by Ψ is

$$X(\Psi) := \{\psi_{j,k} = 2^{j/2}\psi\left(2^j \cdot -k\right) : \psi \in \Psi, j,k \in \mathbb{Z}\}.$$

Analysis is the map

$$T^* : f \mapsto \{\langle f, \psi_{j,k}\rangle : \psi \in \Psi, j,k \in \mathbb{Z}\}, \quad f : \mathbb{R} \to \mathbb{C}.$$

$X(\Psi)$ is a **(tight) frame** if, $\forall f \in L_2$,

$$\|f\|_{L_2} = \|T^* f\|_{\ell_2} := \left(\sum_{\psi \in \Psi, j,k \in \mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^2 \right)^{1/2}.$$

$X(\Psi)$: non-redundant tight frame $\Rightarrow$ $X(\Psi)$ is orthonormal wavelet

The mathematics behind pyramidal algorithms, Part III: framelet-based characterizations of functions spaces

Theorem 2 (Kyriazis, Nielsen) .
$s \in \mathbb{R}, 0 < p < \infty$, $n > \max\{s, -s, 1/p - 1 - s\}$, *integer.*
$X(\Psi)$ *is frame and:*

$$\Psi \subset C_c^n, \qquad \int t^\alpha \psi(t)dt = 0, \quad \forall 0 \leq \alpha \leq n-1, \forall \psi \in \Psi \tag{2}$$

Then we have $\|f\|_{L_p^s} \approx \|Q_\Psi^s f\|_{L_p}$, *where*

$$Q_\Psi^s f := \left(\sum_{\psi \in \Psi, j,k \in \mathbb{Z}} \left| \langle f, \psi_{j,k} \rangle \, 2^{js} \, \chi_{j,k} \right|^2 \right)^{1/2}.$$

Bad News: (2) is too stringent.

Construction of wavelets and framelets

1. Choose a refinable function $\phi \in L_2$ with the refinement (lowpass) filter h.
2. $V_0 :=$ closed linear span of $(\phi(\cdot - k))_{k\in\mathbb{Z}} \subset L_2$.
3. Choose mother wavelets $\Psi = \{\psi_1, .., \psi_r\} \subset V_0(2\cdot)$. Then

$$\widehat{\psi}_i(2\cdot) = \tau_i \widehat{\phi}, \quad (\tau_i\text{: } 2\pi\text{-periodic})$$

Theorem 3 (R-Shen,1997) *Assume*

$$\widehat{h}\overline{\widehat{h}(\cdot + \nu)} + \sum_{i=1}^{r} \tau_i \overline{\tau_i(\cdot + \nu)} = \begin{cases} 1, & \nu = 0, \\ 0, & \nu = \pi. \end{cases}$$

Then $X(\Psi)$ is a tight frame (framelet).

$(h_i)_{i=1}^r$ from the framelet pyramids are the filters of $(\tau_i)_{i=1}^r$.
Much more needed for orthonormal wavelet constructions:
(e.g., $(\phi(\cdot - k))_k$ is orthonormal)

How to measure the “performance” of framelets

1. ϕ provides approximation order m, if, $\forall f \in W_2^m$,

$$\mathrm{dist}(f, V_n) := \min_{g \in V_n} \|f - g\|_{L_2} = O(2^{-nm}), \quad V_n := V_0(2^n \cdot).$$

2. $X(\Psi)$ has vanishing moments of order m_0,
   if $\forall \psi \in \Psi$, $\widehat{\psi} = O(|\cdot|^{m_0})$ near origin.

3. $X(\Psi)$ provides approximation order m', if, $\forall f \in W_2^{m'}$,

$$\|f - \sum_{\psi \in \Psi, k \in \mathbb{Z}, j < n} \langle f, \psi_{j,k} \rangle \psi_{j,k} \|_{L_2} = O(2^{-nm'}).$$

Theorem 4 (Daubechies-Han-R-Shen, 2003)

$$m' = \min\{m, 2m_0\}$$

- For orthonormal wavelets, $m = m' = m_0$.
- For framelets, m_0 can be as small as $m'/2$.

CAP representations

Choose:

- two refinable functions ϕ_c, ϕ_p with refinement filters h_c, h_p.
- A third (Auxiliary-Alignment) lowpass filter h_a.

Decompose: Fix $f : \mathbb{R} \to \mathbb{C}$.
For all $k, j \in \mathbb{Z}$, define $y_j(k) := 2^{j/2} \langle f, (\phi_c)_{j,k} \rangle$.
The CAP operators are:

$$
\begin{aligned}
C : y &\mapsto (h_c * y)_{\downarrow}, && \text{(Coarsification-Compression)},\\
A : y &\mapsto Ay := h_a * y, && \text{(Alignment)},\\
P : y &\mapsto Py := h_p * (y_{\uparrow}), && \text{(Prediction-subdivision)}.
\end{aligned}
$$

Then $Cy_{j+1} = y_j, \forall j$.

The detail coefficients are:

$$d_j := (A - PAC)y_j = Ay_j - PAy_{j-1}.$$

This is the **CAP representation** with (d_j) the **CAP coefficients**.

$$\begin{array}{ccccccccc}
y_m & \xrightarrow{C} & y_{m-1} & \xrightarrow{C} & y_{m-2} & \cdots\cdots & y_1 & \xrightarrow{C} & y_0 \\
\downarrow{\scriptstyle A-PAC} & & \downarrow{\scriptstyle A-PAC} & & \downarrow{\scriptstyle A-PAC} & & \downarrow{\scriptstyle A-PAC} & & \\
d_m & & d_{m-1} & & d_{m-2} & & d_1 & &
\end{array}$$

y_m is recovered from $y_0, d_1, d_2, \ldots, d_m$ since

$$Ay_1 = d_1 + PAy_0, \quad Ay_2 = d_2 + PAy_1, \quad \ldots, \quad Ay_m = d_m + PAy_{m-1}$$

and deconvolving A from Ay_m.

The case $h_a = \delta$ (i.e. $A = I$) and $h_p = h_c(-\cdot)$ is the pyramidal representation of Burt and Adelson.

The mathematics behind pyramidal algorithms, Part IV: CAP-based characterizations of functions spaces, or the winner takes all

Theorem 5

n_c, n_p *integers*, $n_c > \max\{-s, 1/p - 1 - s\}$, $n_p > s$.
Assume that $\phi_c \in C_c^{n_c}, \phi_p \in C_c^{n_p}$, *and*

$$\widehat{h}_c(\cdot{+}\pi) = O(|\cdot|^{n_c}),\ \widehat{h}_p(\cdot{+}\pi) = O(|\cdot|^{n_p}),\ \widehat{h}_a - \widehat{h}_a(2\cdot)\widehat{h}_c\overline{\widehat{h}_p} = O(|\cdot|^{\max\{n_c,n_p\}}),$$

Then:

$$\|f\|_{L_p^s} \approx \|Q^s_{CAP} f\|_{L_p},$$

where

$$Q^s_{CAP} f := \left(\sum_{j,k} \left| d_{j+1}(k)\, 2^{js}\, \chi_{j,k/2} \right|^2 \right)^{1/2} .$$

- No wavelets, no framelets, zilch.

Summary

Do they

	W	F	CAP
implemented by fast pyramid algorithms ?	✓	✓	✓
provides good function space characterizations ?	✓	✓	✓
avoid mother wavelets ?			✓
very short filters, with no artifacts ?		✓	✓
have simple constructions ?		✓	✓
avoid redundant representations ?	✓		

Wavelet are non-redundant. Caplets are only slightly redundant in high dimensions. Their redundancy is non-essential.

CAMP representations: Compression-Alignment-Modified Prediction

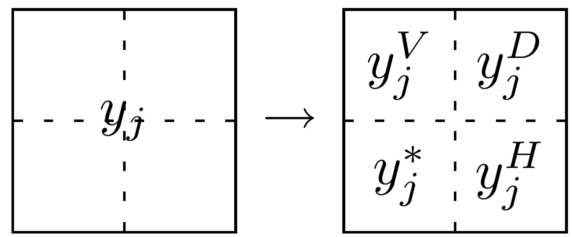
With CAP in hand, one can modify the prediction process s.t.:

- The filters are shorter
- The performance (:= function space characterization) is the same

Example: If h_p is interpolatory, we may redefine the details as:

$$d_j(k) := \begin{cases} (I - PC)y_{j+1}(k), & k \in 2\mathbb{Z}, \\ y_{j+1}(k) - P(y_{j+1\downarrow})(k), & \text{otherwise.} \end{cases}$$

Example (2D): Take $A = I$, $h_c = h_p = [0, \frac{1}{8}, \frac{1}{8}; \frac{1}{8}, \frac{1}{4}, \frac{1}{8}; \frac{1}{8}, \frac{1}{8}, 0] =: h$.
For each y_j, let $y_j^* := (y_{j\downarrow})_\uparrow$ and consider a partition of y_j given as



Define the detail quadruplet $(d_j^*, d_j^H, d_j^V, d_j^D) =: \tilde{d}_j$ as

$$d_j^* := y_j^* - (y_{j-1\uparrow}), \quad d_j^s := y_j^s - Ave_s(y_j^*), \quad s \in \{H, V, D\}.$$

Reconstruction is as before, with a small tweak.

The filters for computing $\tilde{d}_j$ are 4-tap on average: same as 2D Haar.
The performance is much better than Haar.

$\mathbf{y}_j$

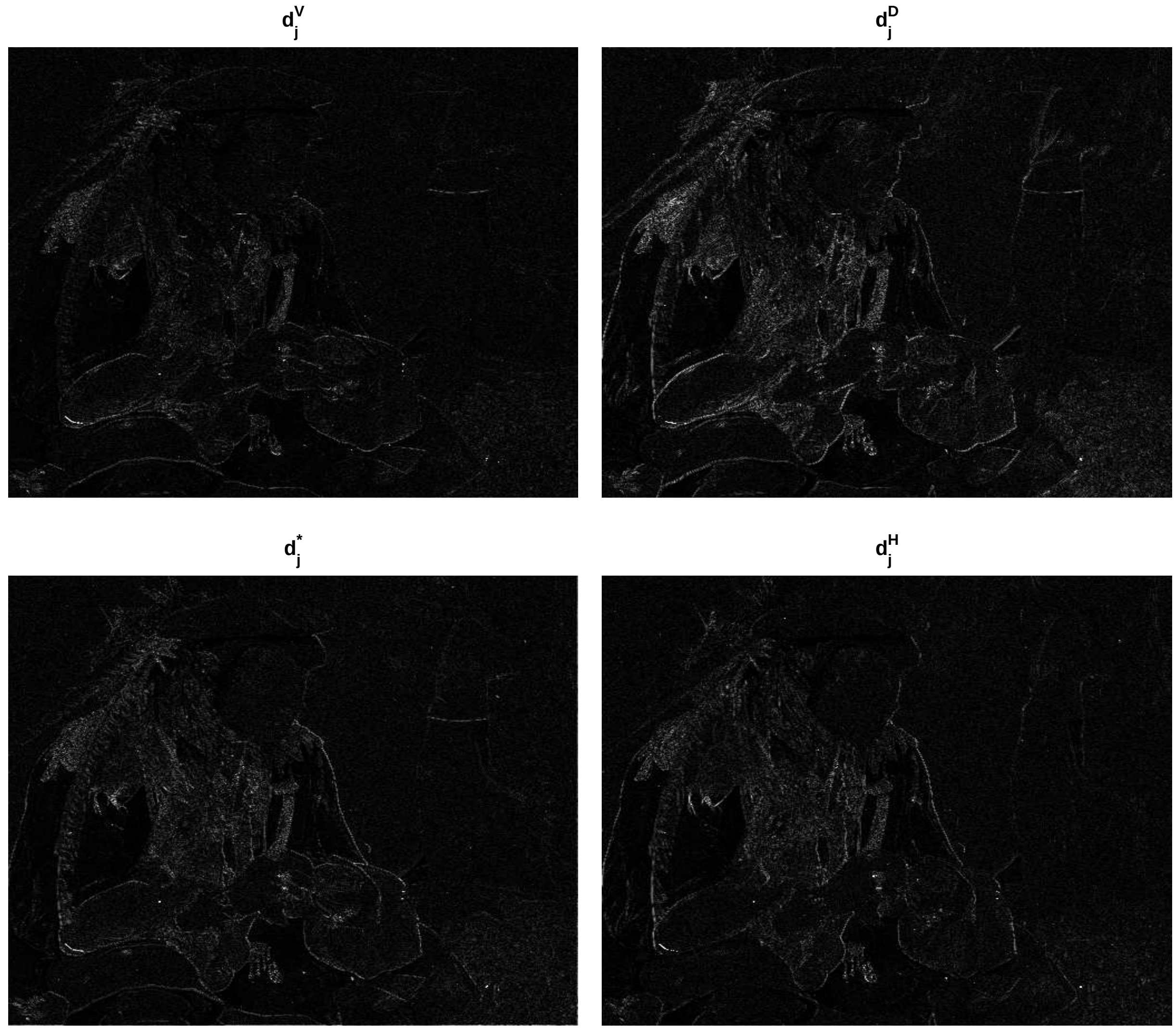


Figure 1: First level $\tilde{d}$ CAMP coefficients, organized by cosets.